Direct transformation of GPS coordinates and baselines into a local coordinate system

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Abstract
For combination with terrestrial measurement methods, GPS-derived coordinates and baselines have to be transformed into a local coordinate system. An alternative using a direct transformation between two Cartesian coordinate systems is described and implemented here. It consists of 1) a preliminary step of computing initial values for the rotation parameters using the Procrustes algorithm, and 2) an eight-parameter transformation that takes different horizontal and vertical scale factors in the local system into account. It does not account for continuously varying scale factors that are the result of conformal mapping or earth curvature and is thus limited to short distances. The approach is illustrated using a numerical example.

Key words: Datum conversion, Procrustes algorithm, Eight-parameter datum transformation

Introduction

GPS baseline vectors and their respective variance-covariance matrices have been firmly established as an observation type in surveying. In order to use GPS baselines not only as slope distances (Kutoglu 2009), but with their full information as 3D vectors, they have to be transformed into the desired local coordinate system.

The traditional method for estimating the necessary datum transformation parameters comprises several steps (Hoffmann-Wellenhoff et al. 1997), as shown in figure Figure 1:

1. Converting local conformal mapping coordinates (n,e) and ellipsoidal heights h, i.e. the triplet (n,e,h), of the common points to geodetic coordinates (λ,φ,h), using the inverse mapping equations for the local coordinate system.

2. Converting geodetic coordinates to geocentric Cartesian coordinates (x,y,z), using the ellipsoidal parameters associated with the geodetic coordinates.

3. Estimating seven similarity transformation parameters $(\Delta x, \Delta y, \Delta z, \alpha, \beta, \gamma, s)$ between the two geocentric Cartesian coordinate systems, using three or more common points in both systems.

Step 1 requires that the inverse mapping equations for converting (n, e) to (φ, λ) must be available to carry out the rigorous conversion in Step 2. If the mapping equations are not available than the triplet (n, e, h) can be used as local Cartesian coordinates within certain approximations. The first and second steps can be omitted if the local Cartesian coordinate system is not tied to the geodetic frame. Such systems may be encountered at construction site surveys and other small-scale engineering projects.

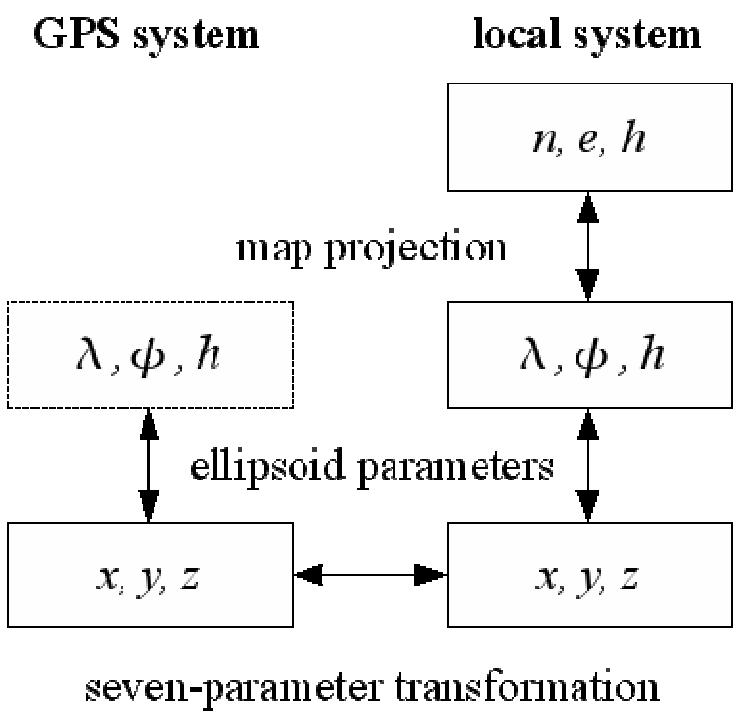

Figure 1: Flowchart of traditional method for datum transformation

The seven parameters required for the transformation are: The three translations (Δx,Δy,Δz) between the center of origin of the two coordinate systems in the translation vector

$$\mathbf{t} = \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} \tag{1}$$

and the three rotation angles (α,β,γ) required for the rotation matrix **R**,

$$\mathbf{R} = \mathbf{R}_3(\gamma)\mathbf{R}_2(\beta)\mathbf{R}_1(\alpha) = \begin{bmatrix} \cos\beta\cos\gamma & \begin{array}{l}\cos\alpha\sin\gamma \\ +\sin\alpha\sin\beta\cos\gamma\end{array} & \begin{array}{l}\sin\alpha\sin\gamma \\ -\cos\alpha\sin\beta\cos\gamma\end{array} \\ -\cos\beta\sin\gamma & \begin{array}{l}\cos\alpha\cos\gamma \\ -\sin\alpha\sin\beta\cos\gamma\end{array} & \begin{array}{l}\sin\alpha\cos\gamma \\ +\cos\alpha\sin\beta\sin\gamma\end{array} \\ \sin\beta & -\sin\alpha\cos\beta & \cos\alpha\cos\beta \end{bmatrix} \tag{2}$$

and the scaling parameter *s*. The transformation is then computed as

$$\mathbf{y}_{1,j} = \mathbf{t} + s\mathbf{R}\mathbf{y}_{2,j} \tag{3}$$

where $\mathbf{y}_{1,j}$ is the vector of coordinates of point *j* in the local (target) system and $\mathbf{y}_{2,j}$ is the vector of coordinates of point *j* in the GPS (source) system.

If geodetic coordinate systems are involved, it is usually assumed that the two global

Cartesian coordinate systems are close to identical (CTI). This makes it possible to replace the rotation matrix **R** in (2) with a simplified version, under the assumption of $\cos\vartheta = 1$ and $\sin\vartheta = \vartheta$:

$$\mathbf{R} = \begin{bmatrix} 1 & \gamma & -\beta \\ -\gamma & 1 & \alpha \\ \beta & -\alpha & 1 \end{bmatrix} \quad (4)$$

Estimating the transformation parameters using a traditional CTI and non-CTI seven-parameter transformation requires iteration due to the non-linearity of the model. . Insufficiently accurate initial parameters might cause diverging iterations. In addition, the local Cartesian coordinates, (n, e, h), may require different scaling for horizontal position and height coordinates, as the horizontal position coordinates are usually the result of a conformal mapping with associated scale variation (Snyder 1987). In order to overcome these problems, a two-step procedure for transforming GPS baselines into local coordinate systems is proposed:

1. Computation of initial values for the rotation parameters (α,β,γ) using the Procrustes algorithm.

2. Refinement of the transformation parameters, including different scaling parameters (s_p, s_h) for horizontal position and height, in a strict non-CTI eight-parameter datum transformation.

The Procrustes Algorithm

The Procrustes algorithm is an effective tool for estimating datum transformation parameters. It does not require initial values for the unknown parameters or an iterative procedure. The use of the Procrustes algorithm for the transformation between two matrices, with optimality in a least-squares sense, dates back to Green (1952) and Schönemann (1966). A recommended starting point for a more detailed introduction to the Procrustes algorithm in the context of datum transformations is Grafarend and Awange (2003). The algorithm has recently been used for datum conversion by (Felus and Burtch 2009).

We use the Procrustes algorithm to compute the initial values for the seven transformation parameters $(\Delta x, \Delta y, \Delta z, \alpha, \beta, \gamma, s)$. More precisely, it computes the translation vector **t** in (1), the rotation matrix **R** in (2) and the scaling parameter *s* for a transformation according to the functional model in (3).

It makes use of $n \geq 3$ common points, where $\mathbf{Y}_1$ is the matrix of coordinates in the target system and $\mathbf{Y}_2$ is the matrix of coordinates in the source system. The quantities **R**, **t**, and *s* are computed according to the following steps:

1. $\mathbf{Y}_1 = \begin{bmatrix} x_1 & y_1 & z_1 \\ \vdots & \vdots & \vdots \\ x_n & y_n & z_n \end{bmatrix}$ and $\mathbf{Y}_2 = \begin{bmatrix} X_1 & Y_1 & Z_1 \\ \vdots & \vdots & \vdots \\ X_n & Y_n & Z_n \end{bmatrix}$

2. centering matrix $\mathbf{C} = \mathbf{I}_n - \frac{1}{n}\mathbf{1}\mathbf{1}^T$, where $\mathbf{I}_n$ is the $n \times n$ Identity matrix,
   and $\mathbf{1}$ is an $n \times 1$ vector of 1's
3. singular value decomposition $\mathbf{Y}_1^T \mathbf{C} \mathbf{Y}_2 = \mathbf{U}\,\mathrm{Diag}(\sigma_1, \sigma_2, \sigma_3)\,\mathbf{V}^T$
4. rotation matrix $\mathbf{R} = \mathbf{U}\mathbf{V}^T$

5. scaling factor $s = \frac{\mathrm{tr}\left(\mathbf{Y}_1^T \mathbf{C} \mathbf{Y}_2 \mathbf{R}^T\right)}{\mathrm{tr}\left(\mathbf{Y}_2^T \mathbf{C} \mathbf{Y}_2\right)}$

6. translation vector $\mathbf{t} = \frac{1}{n}\left(\mathbf{Y}_1 - \mathbf{Y}_2 \mathbf{R}^T s\right)\mathbf{1}$, where again $\mathbf{1}$ is an $n \times 1$ vector of 1's

The values of (α,β,γ) that can be extracted from **R** are then used as initial values in the eight-parameter transformation.

Eight-Parameter Transformation

Using the initial values computed in the previous step, it would be possible to compute a strict non-CTI seven-parameter datum transformation according to the functional model in (3) with the rotation matrix from (2), without the intermediate steps shown in figure 1. This would however neglect, as has been mentioned above, that different scaling may apply for horizontal position and height coordinates. An eighth parameter is introduced to account for this: two different scaling parameters, s_p and s_h, are used for respectively horizontal position and height. This yields an expanded functional model

$$\mathbf{y}_{1,j} = \mathbf{t} + \begin{bmatrix} s_p & & \\ & s_p & \\ & & s_h \end{bmatrix} \mathbf{R}\mathbf{y}_{2,j} \tag{5}$$

with eight unknown parameters: $\mathbf{x} = \left(\Delta x, \Delta y, \Delta z, \alpha, \beta, \gamma, s_p, s_h\right)$.

The initial values for the three angles (α,β,γ) are extracted from the rotation matrix **R** obtained in the previous step:

$$\beta_0 = \arcsin \mathbf{R}_{3,1} \tag{6}$$

$$\alpha_0 = -\frac{\arcsin \mathbf{R}_{2,1}}{\cos \beta_0} \tag{7}$$

$$\gamma_0 = \frac{\arccos \mathbf{R}_{1,1}}{\cos \beta_0} \tag{8}$$

Due to their linear relationship in the functional model, initial values for the other five parameters are not required.

Using the design matrix **A** containing the linearized functional model, the transformation parameters are estimated by iterative least-squares as

$$\hat{\mathbf{x}}_{i+1} = \hat{\mathbf{x}}_i + \left(\mathbf{A}_i^T \mathbf{A}_i\right)^{-1} \mathbf{A}_i^T \mathbf{l}_i \tag{9}$$

with $\hat{\mathbf{x}}_i$ being the result of the previous iteration, and the misclosure vector is

$$\mathbf{l}_i = \begin{bmatrix} \mathbf{l}_{i,1} \\ \vdots \\ \mathbf{l}_{i,n} \end{bmatrix} \tag{10}$$

with

$$\mathbf{l}_{i,j} = \mathbf{y}_{1,j} - \left(\mathbf{t}_i + \begin{bmatrix} s_{p,i} & & \\ & s_{p,i} & \\ & & s_{h,i} \end{bmatrix} \mathbf{R}_i \mathbf{y}_{2,j} \right) \tag{11}$$

where $\mathbf{t}_i$, $\mathbf{R}_i$, $s_{p,i}$, and $s_{h,i}$ are the values of the transformation parameters obtained in the previous iteration. The uterations continue until a suitable convergence criterion is met. Coordinates should be reduced to the centroid of the respective system to avoid numerical instabilities.

Variance/covariance information about the points in the local system $\mathbf{y}_1$ is easily added by means of a weight matrix $\mathbf{P} = \mathbf{C}_{yy}^{-1}$, where $\mathbf{C}_{yy}$ is the covariance matrix for the $\mathbf{y}_1$ coordinates, giving:

$$\hat{\mathbf{x}}_{i+1} = \hat{\mathbf{x}}_i + \left(\mathbf{A}_i^T \mathbf{P} \mathbf{A}_i \right)^{-1} \mathbf{A}_i^T \mathbf{P} \mathbf{l}_i \tag{12}$$

This information can also be added for points in the GPS system $\mathbf{y}_2$ by expanding the vector of observations with the coordinates of these points and expanding the functional model accordingly (Leick 2004).

Numerical Example

In order to show the benefit of using an eight-parameter transformation, a small numerical example is provided. The transformation parameters and the subsequent transformation and residuals have been computed using a non-CTI seven-parameter transformation and eight-parameter transformation described above. The source coordinates were in the WGS84 Cartesian system, the target coordinates in UTM and a local height system. No errors were introduced in the identical points, so an exact transformation is possible except for numerical errors, scale variations caused by the UTM conformal mapping, and neglection of Earth curvature.

Four common points were used (Table 1). In a first step, the initial values for (α, β, γ) were computed using the Procrustes algorithm. These were then used in two least-squares adjustments which estimated or eight transformation parameters respectively. Two iterations were required for the seven-parameter transformation, six for the eight-parameter transformation. A convergence criterion of 1.0e-12 was used for the observation misclosures in (10).

Table 1: Coordinates of common points in meters, in WGS84 geocentric and UTM systems

Point	X	Y	Z	east	north	height
1	3924425.182935	300277.525061	5002122.827517	594445.966528	5760775.553531	40.0
2	3923624.043922	300064.136909	5002772.460568	594274.683825	5761814.439030	50.0
3	3923254.329225	300208.324036	5003001.110433	594438.582878	5762240.084165	10.0
4	3924241.689316	300697.163336	5002215.396208	594874.849982	5760959.579904	20.0

Table 2 lists the residuals after adjustment for both cases. As expected, introducing an additional scaling parameter for the heights significantly reduces all residuals. . It is noted that residuals up to 2.5 mm remain in the horizontal position coordinates. These are a consequence of the conformal mapping condition implied with the UTM projection and cannot be modeled better by just one horizontal scale factor. For larger networks and using ellipsoidal heights the residual in height would become systematic even for the 8-parameter transformation due to the curvature of the ellipsoid. Such large or systematic residuals do not occur when the local target coordinate system is truly projection-free as encountered in many engineering applications.

Table 2: Residuals of common points in millimeters and overall RMS per component.

	7-parameter transformation	8-parameter transformation

Point	r_n	r_e	r_h	r_n	r_e	r_h
1	-0.4	1.3	7.9	-0.8	1.5	0.2
2	0.8	-1.7	-12.6	-0.5	-2.5	-0.1
3	-0.8	1.6	9.5	0.4	1.5	0.1
4	0.3	-1.2	-4.8	0.9	0.5	0.1
RMS	1.2	3.0	18.3	1.4	3.3	0.2

The values and a-posteriori standard deviations of the estimated transformation parameters can be found in table 3. A result of the introduction of the eighth parameter is the significantly improved precision of the translation vector elements and the rotation angles, which appear to absorb the scale inconsistencies. We also note the estimated horizontal scale factor very closely represents the average point scale factor of the UTM projection for the area.

Table 3: Values and standard deviations of estimated transformation parameters.

	7-parameter transformation		8-parameter transformation	
parameter	value	standard deviation	value	standard deviation
α [rad]	-0.05955883	$3 \cdot 10^{-5}$	-0.05947360	$1 \cdot 10^{-5}$
β [rad]	0.66102242	$9 \cdot 10^{-6}$	0.66104844	$3 \cdot 10^{-6}$
γ [rad]	1.64868864	$2 \cdot 10^{-5}$	1.64863665	$6 \cdot 10^{-6}$
Δx [mm]	5941129496	4.2	5936732874	0.9
Δy [mm]	57822114538	4.2	57820796705	0.9
Δz [mm]	-63629935763	4.2	-63563046747	0.9
s [ ]	0.99970552	$6 \cdot 10^{-6}$		
s_p [ ]			0.99970615	$1 \cdot 10^{-6}$
s_h [ ]			0.99865455	$1 \cdot 10^{-4}$

Conclusions

The method described and implemented here allows for the direct transformation of GPS coordinates into a local coordinate system using a two-step approach. The first step is required only for providing initial values for the rotation parameters, guaranteeing convergence of the iterative computation of the second step. The local coordinate system does not need to be tied to a global coordinate system, which makes it possible to use the full 3D information of GPS baselines even for small surveying networks.

The functional model for the non-CTI coordinate transformation has been expanded with an eighth transformation parameter, a separate scaling parameter for heights. The numerical example showed that this additional parameter may be required to achieve sufficient accuracy in local coordinate systems that have differing horizontal and vertical scales, such as is the case with the UTM system and local height systems.

Conformal mapping leads to continuously varying horizontal scale factors. Since the method implemented here estimates only one constant horizontal scale factor, it is limited to distances between the points where the scale variations are smaller than the required precision. In the case of UTM, distances need to be smaller than 5 km if sub-centimeter precision is desired.

Matlab source code for both transformation steps is available, along with documentation and a sample data set. The files are posted on the author's web site, http://www.wittwer.nl.

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